Chapter 1

Vectors

INTRODUCTION

Differential geometry is the study of geometric figures using the methods of calculus. In particular the introductory theory investigates curves and surfaces embedded in three dimensional Euclidean space E^3.

Properties of curves and surfaces which depend only upon points close to a particular point of the figure are called *local* properties. The study of local properties is called differential geometry *in the small.* Those properties which involve the entire geometric figure are called *global* properties. The study of global properties, in particular as they relate to local properties, is called differential geometry *in the large.*

Example 1.1.

Let Q and R be two points near a point P on a curve Γ in a plane and let C_{QR} be the circle through P, Q and R, as shown in Fig. 1-1. Now consider the limiting position of the circles C_{QR} as Q and R approach P. In general, the limiting position will be a circle C tangent to Γ at P. The radius of C is the *radius of curvature* of Γ at P. The radius of curvature is an example of a local property of the curve, for it depends only on the points on Γ near P.

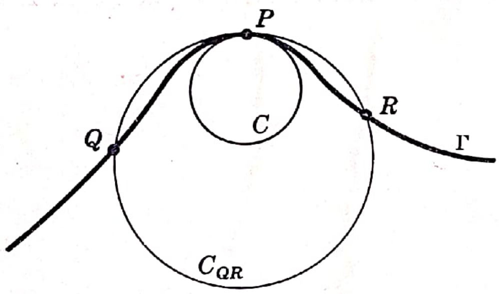

Fig. 1-1

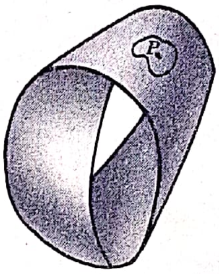

Fig. 1-2

Example 1.2.

The Moebius strip shown in Fig. 1-2 is an example of a one-sided surface. One-sidedness is an example of a global property of a figure, for it depends on the nature of the entire surface. Observe that a small part of the surface surrounding an arbitrary point P is a regular two-sided surface, i.e. *locally* the Moebius strip is two-sided.

We first investigate local properties of curves and surfaces and then apply the results to problems of differential geometry in the large. We begin with a review of vectors in E^3.

VECTORS

By Euclidean space E^3 we mean the set of ordered triplets $\mathbf{a} = (a_1, a_2, a_3)$ with a_1, a_2, a_3 real. A *vector* is a point in E^3 and in general will be denoted by $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}, \mathbf{y}, \ldots$ or $P, Q, R, \ldots$. The *negative* of a vector $\mathbf{a}$ is the vector $-\mathbf{a}$ defined by $-\mathbf{a} = (-a_1, -a_2, -a_3)$. The *zero* vector is the vector $\mathbf{0} = (0, 0, 0)$. The *length* or *magnitude* of a vector $\mathbf{a} = (a_1, a_2, a_3)$ is the real number $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. Clearly $|\mathbf{a}| \geqq 0$ and $|\mathbf{a}| = 0$ if and only if $\mathbf{a} = \mathbf{0}$.

ADDITION OF VECTORS

Given two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in E^3, their *sum* $\mathbf{a}+\mathbf{b}$ is the vector defined by

$$\mathbf{a}+\mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

The difference of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$. In Problem 1.1 we prove that vector addition satisfies

[$\mathbf{A}_1$] $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$ (Commutative Law)

[$\mathbf{A}_2$] $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$ (Associative Law)

[$\mathbf{A}_3$] $\mathbf{0}+\mathbf{a} = \mathbf{a}$ for all $\mathbf{a}$

[$\mathbf{A}_4$] $\mathbf{a}+(-\mathbf{a}) = \mathbf{0}$ for all $\mathbf{a}$

Example 1.3.

Let $\mathbf{a} = (1,-2,0)$ and $\mathbf{b} = (0,1,1)$. Then $\mathbf{a}+\mathbf{b} = (1,-1,1)$, $-\mathbf{a} = (-1,2,0)$, $\mathbf{b}-\mathbf{a} = (-1,3,1)$, $|\mathbf{a}| = \sqrt{5}$.

Example 1.4.

Using [$\mathbf{A}_1$] through [$\mathbf{A}_4$] we see that for any $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{a}+(\mathbf{b}-\mathbf{a}) = \mathbf{a}+(\mathbf{b}+(-\mathbf{a})) = \mathbf{a}+(-\mathbf{a})+\mathbf{b} = \mathbf{0}+\mathbf{b} = \mathbf{b}$$

Thus the vector equation $\mathbf{a}+\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} = \mathbf{b}-\mathbf{a}$. It is also the only solution. For if $\mathbf{a}+\mathbf{y} = \mathbf{b}$, then

$$(-\mathbf{a})+\mathbf{a}+\mathbf{y} = (-\mathbf{a})+\mathbf{b} = \mathbf{b}-\mathbf{a}, \quad\text{or}\quad \mathbf{0}+\mathbf{y} = \mathbf{b}-\mathbf{a}, \quad\text{or}\quad \mathbf{y} = \mathbf{b}-\mathbf{a}$$

Given two points P and Q in E^3 (that is, two vectors P and Q) we introduce the special notation $\mathbf{PQ}$ for their difference $Q-P$ and we picture $\mathbf{PQ}$ as an arrow drawn from P to Q as shown in Fig. 1-3. By the distance from P to Q we mean the length $|\mathbf{PQ}|$. Evidently $\mathbf{PQ} = -\mathbf{QP}$, $|\mathbf{PQ}| = |\mathbf{QP}|$, $\mathbf{PQ} = \mathbf{P'Q'}$ if and only if $Q-P = Q'-P'$, and $\mathbf{PP} = \mathbf{0}$ for all P.

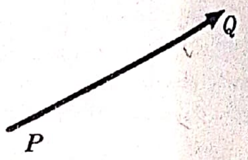

Fig. 1-3

Example 1.5.

Let $\mathbf{a} = \mathbf{PQ}$, $\mathbf{b} = \mathbf{QR}$ and $\mathbf{c} = \mathbf{RS}$, $\mathbf{d} = \mathbf{SP}$ as shown in Fig. 1-4. Then

$$\mathbf{a}+\mathbf{b} = \mathbf{PQ}+\mathbf{QR} = Q-P+R-Q = R-P = \mathbf{PR}$$

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{PR}+\mathbf{RS} = R-P+S-R = S-P = \mathbf{PS} = -\mathbf{d}$$

$$\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d} = \mathbf{PS}+\mathbf{SP} = S-P+P-S = \mathbf{0}$$

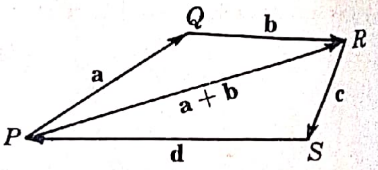

Fig. 1-4

MULTIPLICATION OF A VECTOR BY A SCALAR

If k is a real number and $\mathbf{a} = (a_1, a_2, a_3)$ a vector, we define the product $k\mathbf{a}$ to be the vector

$$k\mathbf{a} = (ka_1, ka_2, ka_3)$$

Clearly $0\mathbf{a} = k\mathbf{0} = \mathbf{0}$ for all k and $\mathbf{a}$.

In the study of vectors we usually refer to the real numbers as *scalars*. The product $k\mathbf{a}$ is called *multiplication of a vector by a scalar.*

In Problem 1.4 we prove that multiplication of vectors by scalars satisfies

[$\mathbf{B}_1$] $k_1(k_2\mathbf{a}) = (k_1k_2)\mathbf{a} = k_1k_2\mathbf{a}$

[$\mathbf{B}_2$] $(k_1+k_2)\mathbf{a} = k_1\mathbf{a}+k_2\mathbf{a}$; $k(\mathbf{a}+\mathbf{b}) = k\mathbf{a}+k\mathbf{b}$ (Distributive Laws)

[$\mathbf{B}_3$] $1\mathbf{a} = \mathbf{a}$

Finally, if $\mathbf{a} = (a_1, a_2, a_3)$, then

$$|k\mathbf{a}| = \sqrt{(ka_1)^2 + (ka_2)^2 + (ka_3)^2} = \sqrt{k^2}\sqrt{a_1^2 + a_2^2 + a_3^2}$$

Thus for all k and $\mathbf{a}$,

$$|k\mathbf{a}| = |k|\,|\mathbf{a}| \tag{1.1}$$

Example 1.6.

Let $\mathbf{a} = (1, \pi, 0)$ and $\mathbf{b} = (0, 2, -1)$. Then $2\mathbf{a} = (2, 2\pi, 0)$, $(-1)\mathbf{a} = (-1, -\pi, 0) = -\mathbf{a}$, and $\mathbf{a} - 3\mathbf{b} = (1, \pi - 6, 3)$.

Example 1.7.

Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be given vectors and let $\mathbf{a} = \mathbf{u}_1 - 2\mathbf{u}_2$, $\mathbf{b} = -\mathbf{u}_2 + 2\mathbf{u}_3$, and $\mathbf{c} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$. Then

$$\begin{aligned} \mathbf{a} - 2\mathbf{b} - \mathbf{c} &= (\mathbf{u}_1 - 2\mathbf{u}_2) - 2(-\mathbf{u}_2 + 2\mathbf{u}_3) - (\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) \\ &= \mathbf{u}_1 - 2\mathbf{u}_2 + 2\mathbf{u}_2 - 4\mathbf{u}_3 - \mathbf{u}_1 - \mathbf{u}_2 - \mathbf{u}_3 = -\mathbf{u}_2 - 5\mathbf{u}_3 \end{aligned}$$

A vector $\mathbf{a}$ is said to have the same *direction* as a nonzero vector $\mathbf{b}$ if for some $k \geqq 0$, $\mathbf{a} = k\mathbf{b}$. If $\mathbf{a}$ has the same direction as $\mathbf{b}$ and also the same length as $\mathbf{b}$, then from equation (*1.1*), $|\mathbf{a}| = |k|\,|\mathbf{b}| = k|\mathbf{b}| = |\mathbf{b}|$. Thus $k = 1$, and $\mathbf{a}$ equals $\mathbf{b}$. A vector is thus uniquely determined by its direction and length. If $\mathbf{a} = k\mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$ and $k \leqq 0$, then $\mathbf{a}$ has the *opposite direction* to $\mathbf{b}$. If $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ has the same or opposite direction to $\mathbf{b}$, i.e. $\mathbf{a} = k\mathbf{b}$ for some real k, then $\mathbf{a}$ is *parallel* to $\mathbf{b}$.

We call a vector $\mathbf{u}$ of unit length a *unit* vector. In general $\mathbf{u}_\mathbf{a}$ shall denote the unit vector in the direction of a nonzero vector $\mathbf{a}$. Clearly this is obtained by multiplying $\mathbf{a}$ by $1/|\mathbf{a}|$, i.e.

$$\mathbf{u}_\mathbf{a} = \mathbf{a}/|\mathbf{a}| \tag{1.2}$$

Example 1.8.

Let $\mathbf{a} = (1, -1, 3)$, $\mathbf{b} = (2, -2, 6)$ and $\mathbf{c} = (-3, 3, -9)$. Since $\mathbf{a} = \frac{1}{2}\mathbf{b}$, the vectors $\mathbf{a}$ and $\mathbf{b}$ have the same direction. The vectors $\mathbf{b}$ and $\mathbf{c}$ have opposite directions since $\mathbf{b} = -(2/3)\mathbf{c}$. The unit vector in the direction of $\mathbf{a}$ is the vector $\mathbf{u}_\mathbf{a} = \mathbf{a}/|\mathbf{a}| = (1/\sqrt{11}, -1/\sqrt{11}, 3/\sqrt{11})$.

Example 1.9.

In the triangle OAB shown in Fig. 1-5, let $\mathbf{a} = \mathbf{OA}$ and $\mathbf{b} = \mathbf{OB}$, and let M be the midpoint of side AB. Then the vector $\mathbf{OM}$ can be expressed in terms of $\mathbf{a}$ and $\mathbf{b}$ as follows:

$$\begin{aligned} \mathbf{OM} &= \mathbf{a} + \mathbf{AM} = \mathbf{a} + \tfrac{1}{2}\mathbf{AB} \\ &= \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) = \mathbf{a} + \tfrac{1}{2}\mathbf{b} - \tfrac{1}{2}\mathbf{a} \\ &= \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b} \end{aligned}$$

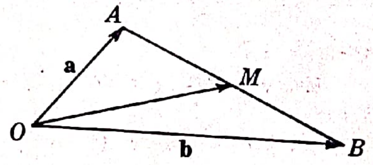

Fig. 1-5

LINEAR DEPENDENCE AND INDEPENDENCE

We now define the very important concepts of linear dependence and linear independence. Namely, the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are said to be *linearly dependent* if there exist scalars $k_1, k_2, \ldots, k_n$ *not all zero* such that

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n = \mathbf{0} \tag{1.3}$$

The vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are said to be *linearly independent* if they are not linearly dependent. That is, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent if (*1.3*) implies all $k_1 = k_2 = \cdots = k_n = 0$.

Note that a set of vectors which includes the zero vector is dependent; for we can always write $1\mathbf{0} + 0\mathbf{u}_1 + \cdots + 0\mathbf{u}_n = \mathbf{0}$.

Example 1.10.

The vectors $\mathbf{a} = (1, -1, 0)$, $\mathbf{b} = (0, 2, -1)$, $\mathbf{c} = (2, 0, -1)$ are linearly dependent, since $2\mathbf{a} + \mathbf{b} - \mathbf{c} = \mathbf{0}$.

Example 1.11.

Suppose $\mathbf{a}$ is parallel to $\mathbf{b}$. Then $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or $\mathbf{a} = k\mathbf{b}$, i.e. $\mathbf{a} - k\mathbf{b} = \mathbf{0}$. Thus $\mathbf{a}$ and $\mathbf{b}$ are dependent. Conversely, suppose $\mathbf{a}$ and $\mathbf{b}$ are dependent. Then $k_1\mathbf{a} + k_2\mathbf{b} = \mathbf{0}$ where, say, $k_1 \neq 0$. But then $\mathbf{a} = -(k_2/k_1)\mathbf{b}$. Thus two vectors are dependent if and only if they are parallel.

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In Problem 1.10 we prove the following important property of linearly *independent* vectors:

Theorem 1.1. If a vector is expressed as a linear function of independent vectors, then it is expressed so *uniquely*. That is, if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are independent, and if

$$\mathbf{u} = k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n = k_1'\mathbf{u}_1 + k_2'\mathbf{u}_2 + \cdots + k_n'\mathbf{u}_n$$

then $k_1 = k_1',\ k_2 = k_2',\ \ldots,\ k_n = k_n'$.

BASES AND COMPONENTS

The three vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$ and $\mathbf{e}_3 = (0,0,1)$ are independent. For $k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3 = (k_1, k_2, k_3)$ and so if $k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3 = \mathbf{0}$, then $k_1 = k_2 = k_3 = 0$. Also any vector $\mathbf{a} = (a_1, a_2, a_3)$ can be written $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ as a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, and by Theorem 1.1 this representation is unique.

In general we call a set of vectors B a *basis* for E^3 if (i) *every* vector in E^3 can be written as a linear combination of the vectors in B, (ii) B is a linearly independent set of vectors.

In Problem 1.11 we prove

Theorem 1.2. Any three linearly independent vectors form a basis in E^3. Conversely, every basis in E^3 consists of three linearly independent vectors.

Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis in space and let $\mathbf{a} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$. The scalars a_1, a_2, a_3, for short, a_i, $i = 1,2,3$, are called the *components* of $\mathbf{a}$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

It follows from Theorem 1.1 that the components of a vector with respect to a given basis are unique. However, note that the components of a vector depend upon the basis chosen and in general the components will change if there is a change in basis. An exception is the vector $\mathbf{0}$ whose components are always 0, 0, 0.

In general we shall denote the components of vectors $\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \mathbf{u}, \ldots$ with respect to some prescribed basis by $a_i, b_i, x_i, y_i, u_i, \ldots$.

Example 1.12.

Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and let $\mathbf{a} = 2\mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{b} = \mathbf{u}_2 - 2\mathbf{u}_3$, and $\mathbf{c} = 3\mathbf{u}_1 + \mathbf{u}_3$. We will show that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent and hence also form a basis. For, suppose

$$k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c} = (2k_1 + 3k_3)\mathbf{u}_1 + (-k_1 + k_2)\mathbf{u}_2 + (-2k_2 + k_3)\mathbf{u}_3 = \mathbf{0}$$

Since the $\mathbf{u}_i$ are independent, it follows that

$$2k_1 + 3k_3 = 0, \quad -k_1 + k_2 = 0, \quad -2k_2 + k_3 = 0$$

This is a system of three homogeneous linear equations in k_1, k_2, k_3. Since the determinant of the coefficients

$$\det\begin{pmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = 8 \neq 0$$

the *only* solution is $k_1 = k_2 = k_3 = 0$. Hence the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are independent. Observe that the components of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ appear as the columns in the above determinant.

As suggested in the above example, we have in general

Theorem 1.3. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and let

$$\begin{aligned} \mathbf{v}_1 &= a_{11}\mathbf{u}_1 + a_{21}\mathbf{u}_2 + a_{31}\mathbf{u}_3 \\ \mathbf{v}_2 &= a_{12}\mathbf{u}_1 + a_{22}\mathbf{u}_2 + a_{32}\mathbf{u}_3 \\ \mathbf{v}_3 &= a_{13}\mathbf{u}_1 + a_{23}\mathbf{u}_2 + a_{33}\mathbf{u}_3 \end{aligned}$$

or, in short, $\mathbf{v}_j = \sum_{i=1}^{3} a_{ij}\mathbf{u}_i,\ j = 1, 2, 3$. Then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis if and only if

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \neq 0$$

SCALAR PRODUCT OF VECTORS

The *dot* or *scalar product* of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is the real number

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

In particular, for $\mathbf{a} = \mathbf{b}$ we have the formula

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \qquad (1.4)$$

In Problem 1.14 we prove that scalar multiplication satisfies

[**C**$_1$] $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (Symmetric Law)

[**C**$_2$] $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$ $(k = \text{scalar})$

[**C**$_3$] $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (Distributive Law)

[**C**$_4$] Scalar multiplication is *positive definite*; that is,

(i) $\mathbf{a} \cdot \mathbf{a} \geqq 0$ for all $\mathbf{a}$

(ii) $\mathbf{a} \cdot \mathbf{a} = 0$ if and only if $\mathbf{a} = 0$

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Clearly, from the definition, $\mathbf{a} \cdot 0 = 0$ for all $\mathbf{a}$. Also, if $\mathbf{a} \cdot \mathbf{b} = 0$ for all $\mathbf{a}$, then $\mathbf{b} \cdot \mathbf{b} = 0$, and hence from [**C**$_4$](ii), $\mathbf{b} = 0$.

Example 1.13.

Let $\mathbf{a} = (-2, 1, 0)$ and $\mathbf{b} = (2, 1, 1)$. Then $\mathbf{a} \cdot \mathbf{b} = -3$ and $\mathbf{a} \cdot \mathbf{a} = 5 = |\mathbf{a}|^2$.

Example 1.14.

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be given vectors and let $\mathbf{a} = \mathbf{u}_1 - \mathbf{u}_2$ and $\mathbf{b} = 2\mathbf{u}_1 + \mathbf{u}_2$. Then

$$\mathbf{a} \cdot \mathbf{b} = (\mathbf{u}_1 - \mathbf{u}_2) \cdot (2\mathbf{u}_1 + \mathbf{u}_2) = 2\mathbf{u}_1 \cdot \mathbf{u}_1 - 2\mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{u}_1 \cdot \mathbf{u}_2 - \mathbf{u}_2 \cdot \mathbf{u}_2 = 2|\mathbf{u}_1|^2 - \mathbf{u}_1 \cdot \mathbf{u}_2 - |\mathbf{u}_2|^2$$

In Problem 1.16 we prove the *Cauchy-Schwarz inequality*

$$|\mathbf{a} \cdot \mathbf{b}| \leqq |\mathbf{a}|\,|\mathbf{b}|$$

with *equality holding if and only if* **a** *and* **b** *are linearly dependent.* The *angle between two nonzero vectors* **a** *and* **b**, denoted by $\theta = \measuredangle(\mathbf{a}, \mathbf{b})$, is the unique solution of

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta \qquad (1.5)$$

satisfying $0 \leqq \theta \leqq \pi$.

Example 1.15.

In the triangle ABC shown in Fig. 1-6, let $\mathbf{a} = \mathbf{BC}$, $\mathbf{b} = \mathbf{AC}$, $\mathbf{c} = \mathbf{BA} = \mathbf{a} - \mathbf{b}$, and $\theta = \measuredangle ACB = \measuredangle(\mathbf{a}, \mathbf{b})$. If we consider

$$|\mathbf{c}|^2 = |\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$$

we have the law of cosines

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta + |\mathbf{b}|^2$$

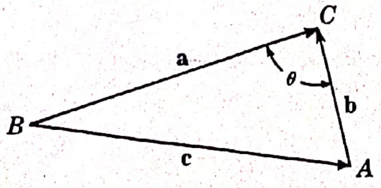


Fig. 1-6

Let **b** be a nonzero vector. The *scalar projection of* **a** *onto* **b**, denoted by $P_{\mathbf{b}}(\mathbf{a})$, is the scalar $P_{\mathbf{b}}(\mathbf{a}) \equiv (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|$. The vector $P_{\mathbf{b}}(\mathbf{a})\mathbf{u}_{\mathbf{b}}$, where $\mathbf{u}_{\mathbf{b}}$ is the *unit vector in the direction of* **b**, is called the *vector projection of* **a** *onto* **b** and is denoted by $\mathbf{P}_{\mathbf{b}}(\mathbf{a})$. It follows that

$$\mathbf{P}_{\mathbf{b}}(\mathbf{a}) = P_{\mathbf{b}}(\mathbf{a})\mathbf{u}_{\mathbf{b}} = ((\mathbf{a}\cdot\mathbf{b})/|\mathbf{b}|)(\mathbf{b}/|\mathbf{b}|) = \frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}$$

Clearly $P_{\mathbf{b}}(0) = 0$ and $\mathbf{P}_{\mathbf{b}}(0) = 0$. If $\mathbf{a} \neq 0$, then from equation (*1.5*), $P_{\mathbf{b}}(\mathbf{a}) = |\mathbf{a}|$ and $\mathbf{P}_{\mathbf{b}}(\mathbf{a}) = |\mathbf{a}| \cos\theta\,\mathbf{u}_{\mathbf{b}}$, where $\theta = \measuredangle(\mathbf{a},\mathbf{b})$. It follows that $P_{\mathbf{b}}(\mathbf{a})$ and $\mathbf{P}_{\mathbf{b}}(\mathbf{a})$ are indepen of the length of **b** but depend only on its direction as indicated in Fig. 1-7. In fact, vector $\mathbf{P}_{\mathbf{b}}(\mathbf{a})$ is also independent of the sense of **b**; that is, $\mathbf{P}_{-\mathbf{b}}(\mathbf{a}) = \mathbf{P}_{\mathbf{b}}(\mathbf{a})$. For

$$\mathbf{P}_{-\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a}\cdot(-\mathbf{b})}{|-\mathbf{b}|^2}(-\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \mathbf{P}_{\mathbf{b}}(\mathbf{a})$$

The scalar $P_{\mathbf{b}}(\mathbf{a})$ changes sign with a change in the sense of **b**.

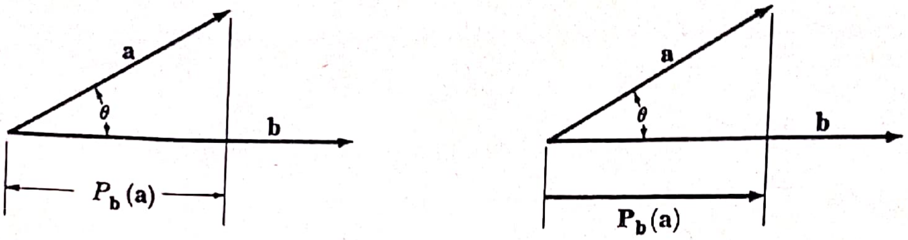

a
θ
b
$P_{\mathbf{b}}(\mathbf{a})$
a
θ
b
$\mathbf{P}_{\mathbf{b}}(\mathbf{a})$

Fig. 1-7

ORTHOGONAL VECTORS

Two vectors **a** and **b** are said to be orthogonal, written $\mathbf{a} \perp \mathbf{b}$, if $\mathbf{a}\cdot\mathbf{b} = 0$. It follows from equation (*1.5*) that **a** and **b** are orthogonal if and only if either $\mathbf{a} = 0$, $\mathbf{b} = 0$ or $\theta = \measuredangle(\mathbf{a},\mathbf{b}) = \pi/2$.

Example 1.16.

Let **a** and **b** be linearly independent and let $\mathbf{c} = \mathbf{a} - \mathbf{P}_{\mathbf{b}}(\mathbf{a})$. Then **c** is a nonzero vector orthogonal to **b**. For suppose $\mathbf{c} = 0$; then from equation (*1.6*), $0 = 1\mathbf{a} - \mathbf{P}_{\mathbf{b}}(\mathbf{a}) = 1\mathbf{a} - k\mathbf{b}$, where $k = (\mathbf{a}\cdot\mathbf{b})/|\mathbf{b}|^2$, which is impossible since **a** and **b** are independent. Hence $\mathbf{c} \neq 0$. Finally,

$$\mathbf{c}\cdot\mathbf{b} = \left(\mathbf{a} - \frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}\right)\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b} - \frac{(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{b})}{|\mathbf{b}|^2} = (\mathbf{a}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b}) = 0$$

Thus $\mathbf{c} \perp \mathbf{b}$.

ORTHONORMAL BASES

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be three mutually orthogonal unit vectors as shown in Fig. 1-8. These vectors are independent; for if $k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3 = 0$, then $0 = \mathbf{e}_i\cdot 0 = \mathbf{e}_i\cdot(k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3) = \mathbf{e}_i\cdot k_i\mathbf{e}_i = k_i$ or $k_i = 0$ for each i. Therefore they form a basis called an *orthonormal* basis.

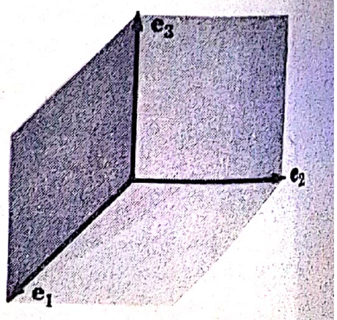


Fig. 1-8

We observe that $\mathbf{e}_i$, $i = 1, 2, 3$, is an orthonormal basis if and only if

$$\mathbf{e}_1\cdot\mathbf{e}_1 = \mathbf{e}_2\cdot\mathbf{e}_2 = \mathbf{e}_3\cdot\mathbf{e}_3 = 1 \quad \text{(Unit vectors)}$$
$$\mathbf{e}_1\cdot\mathbf{e}_2 = \mathbf{e}_2\cdot\mathbf{e}_3 = \mathbf{e}_1\cdot\mathbf{e}_3 = 0 \quad \text{(Mutually orthogonal)}$$

or, in short,

$$\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases} \qquad (i, j = 1, 2, 3) \tag{1.7}$$

The quantity δ_{ij} is called the *Kronecker* symbol and will be used repeatedly.

In Problem 1.23 we prove

Theorem 1.4. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis and let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. Then

$$\text{(i)}\quad \mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \sum_{i=1}^{3} a_ib_i$$

$$\text{(ii)}\quad |\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\sum_{i=1}^{3} a_i^2}$$

$$\text{(iii)}\quad a_i = \mathbf{a}\cdot\mathbf{e}_i, \quad (i = 1, 2, 3).$$

Example 1.17.

Let $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_3$, $\mathbf{b} = 2\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$, and $\mathbf{c} = -2\mathbf{e}_2 + \mathbf{e}_3$. Then

(*a*) $\mathbf{a}\cdot\mathbf{b} = (1)(2) + (0)(1) + (2)(-2) = -2$

(*b*) $(\mathbf{a}\cdot\mathbf{c})\mathbf{b} = [(1)(0) + (0)(-2) + (2)(1)]\,(2\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3) = 4\mathbf{e}_1 + 2\mathbf{e}_2 - 4\mathbf{e}_3$

(*c*) $|\mathbf{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

(*d*) $\mathbf{u}_\mathbf{a} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = (1/\sqrt{5})\mathbf{e}_1 + (2/\sqrt{5})\mathbf{e}_3$

(*e*) $\cos \measuredangle(\mathbf{a}, \mathbf{b}) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-2}{3\sqrt{5}}$

Let a nonzero vector $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and let $\theta_i = \measuredangle(\mathbf{a}, \mathbf{e}_i)$, $i = 1, 2, 3$, as shown in Fig. 1-9. The scalars $\cos\theta_1, \cos\theta_2, \cos\theta_3$ are called the *direction cosines* of $\mathbf{a}$. Since $\mathbf{a}\cdot\mathbf{e}_i = |\mathbf{a}|\cos\theta_i = a_i$, we have

$$\cos\theta_i = a_i/|\mathbf{a}|, \quad i = 1, 2, 3$$

Note that

$$\mathbf{u}_\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{a_1}{|\mathbf{a}|}\mathbf{e}_1 + \frac{a_2}{|\mathbf{a}|}\mathbf{e}_2 + \frac{a_3}{|\mathbf{a}|}\mathbf{e}_3$$
$$= (\cos\theta_1)\mathbf{e}_1 + (\cos\theta_2)\mathbf{e}_2 + (\cos\theta_3)\mathbf{e}_3$$

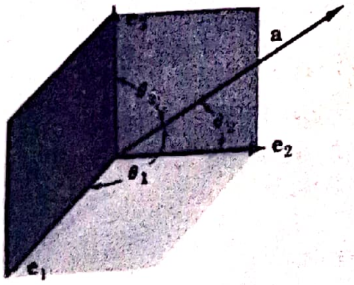


Fig. 1-9

That is, the direction cosines of $\mathbf{a}$ are the components of the unit vector in the direction of $\mathbf{a}$.

ORIENTED BASES

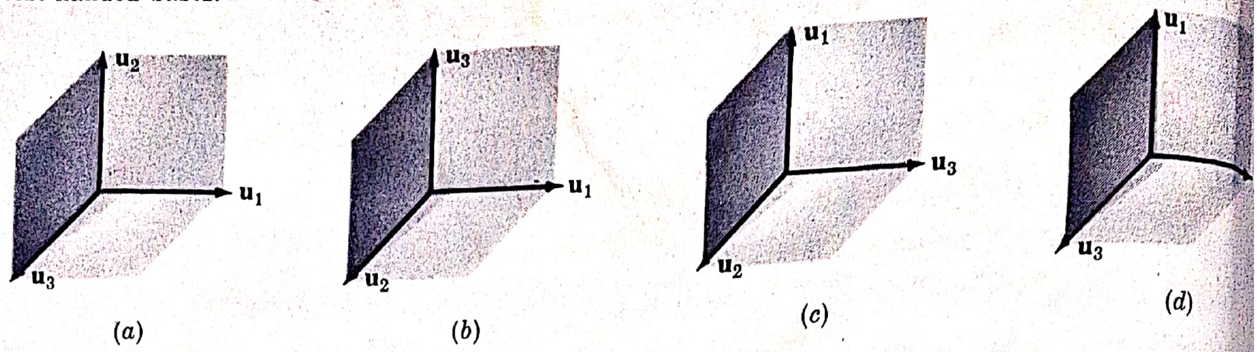
Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ be *ordered* orthonormal bases and imagine that the triad $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ is rotated to make $\mathbf{g}_1$ and $\mathbf{g}_2$ coincide with $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively. Then $\mathbf{g}_3$ will either coincide with $\mathbf{e}_3$ in which case we say that $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ has the same orientation as $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, or $\mathbf{g}_3$ will point in the direction opposite to $\mathbf{e}_3$, in which case the bases are said to have *opposite orientation*. To formulate this concept of orientation in a precise manner, not only for orthonormal bases but for arbitrary bases, we proceed as follows:

Let $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ be ordered bases and let $\mathbf{v}_j = \sum_{i=1}^{3} a_{ij}\mathbf{u}_i$. Then $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ if $\det(a_{ij}) > 0$. In Problem 1.27 we show that this defines an equivalence relation on the set of all ordered bases in E^3. This relation partitions the bases into exactly two equivalence classes. Ordered bases in the same class have the same orientation and ordered bases in different classes have opposite orientation.

In order to distinguish graphically one orientation of an ordered basis, we say $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is a *right-handed* basis if the vectors assume the same directions in space as the thumb, index finger and middle finger of the right hand; otherwise the basis is said to be a *left-handed system*.

Example 1.18.

The triplets $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ in Fig. 1-10(*a*) and (*c*) are right-handed bases. In Fig. 1-10(*b*) and (*d*) th left-handed bases.



(*a*) (*b*) (*c*) (*d*)

Fig. 1-10

Note. Unless stated otherwise, our bases shall be right-handed orthonormal bases.

VECTOR PRODUCT OF VECTORS

Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a right-handed orthonormal basis and let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ ar $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. The *cross* or *vector* product of **a** and **b**, denoted by $\mathbf{a} \times \mathbf{b}$, is the vecto

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3$$

As an aid in computing the above, we observe that it can be obtained as the expansion of th determinant

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \det\begin{pmatrix} \mathbf{e}_1 & a_1 & b_1 \\ \mathbf{e}_2 & a_2 & b_2 \\ \mathbf{e}_3 & a_3 & b_3 \end{pmatrix} = \mathbf{e}_1 \det\begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} - \mathbf{e}_2 \det\begin{pmatrix} a_1 & b_1 \\ a_3 & b_3 \end{pmatrix} + \mathbf{e}_3 \det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \\ &= (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3\end{aligned}$$

Example 1.19.

Let $\mathbf{a} = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{b} = \mathbf{e}_2 + 2\mathbf{e}_3$, $\mathbf{c} = -2\mathbf{e}_1 - 2\mathbf{e}_3$. Then

$$\mathbf{a} \times \mathbf{b} = \det\begin{pmatrix} \mathbf{e}_1 & 1 & 0 \\ \mathbf{e}_2 & -1 & 1 \\ \mathbf{e}_3 & 0 & 2 \end{pmatrix} = -2\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$$

In Problem 1.32 we prove that the vector product is in fact independent of the right-handed orthonormal basis chosen. Also in Problem 1.31 we prove

Theorem 1.5. (i) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, where $\theta = \measuredangle(\mathbf{a}, \mathbf{b})$

(ii) a. $(\mathbf{a} \times \mathbf{b}) \perp \mathbf{a}$ and $(\mathbf{a} \times \mathbf{b}) \perp \mathbf{b}$

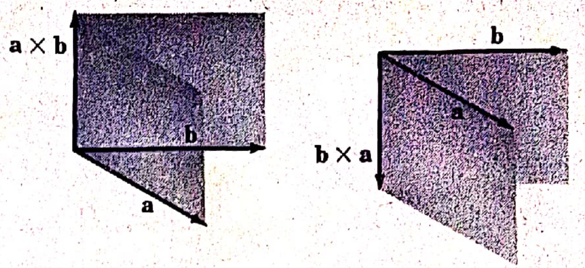
b. If $\mathbf{a} \times \mathbf{b} \neq 0$, then $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ is a right-handed linearly independent triplet.

Since $|\mathbf{a}|\,|\mathbf{b}| \sin\theta = 0$ if and only if $|\mathbf{a}| = 0$, $|\mathbf{b}| = 0$, $\theta = 0$, or $\theta = \pi$, we have from (i) above and the strict form of the Schwarz inequality (that is, $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$ if and only if **a** and **b** are linearly dependent),

Theorem 1.6. $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if **a** and **b** are linearly dependent.

If $\mathbf{a}$ and $\mathbf{b}$ are not linearly dependent, i.e. if $\mathbf{a}\times\mathbf{b}\neq 0$, Theorem 1.5(ii) states that $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$ and to $\mathbf{b}$ and such that $(\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b})$ is a right-handed triplet, as shown in Fig. 1-11(*a*).

Observe that the vector product is in general not commutative. Although $\mathbf{b}\times\mathbf{a}$ has the same magnitude as $\mathbf{a}\times\mathbf{b}$ (Theorem 1.5(i)) and is parallel to $\mathbf{a}\times\mathbf{b}$ (Theorem 1.5(ii)a), it has the opposite direction (Theorem 1.5(ii)b). Thus $\mathbf{b}\times\mathbf{a} = -(\mathbf{a}\times\mathbf{b})$, as shown in Fig. 1.11(*b*).



(*a*) (*b*)

Fig. 1-11

Example 1.20.

For an orthonormal basis $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ shown in Fig. 1-12, it follows from Theorem 1.5 that

$$\begin{array}{lll} \mathbf{g}_1\times\mathbf{g}_1 = 0 & \mathbf{g}_2\times\mathbf{g}_1 = -\mathbf{g}_3 & \mathbf{g}_3\times\mathbf{g}_1 = \mathbf{g}_2 \\ \mathbf{g}_1\times\mathbf{g}_2 = \mathbf{g}_3 & \mathbf{g}_2\times\mathbf{g}_2 = 0 & \mathbf{g}_3\times\mathbf{g}_2 = -\mathbf{g}_1 \\ \mathbf{g}_1\times\mathbf{g}_3 = -\mathbf{g}_2 & \mathbf{g}_2\times\mathbf{g}_3 = \mathbf{g}_1 & \mathbf{g}_3\times\mathbf{g}_3 = 0 \end{array}$$

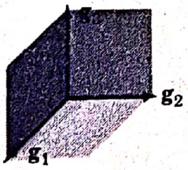


Fig. 1-12

In Problem 1.29 we prove that the vector product satisfies

[E$_1$] $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$ (Anticommutative Law)

[E$_2$] $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ (Distributive Law)

[E$_3$] $(k\mathbf{a})\times\mathbf{b} = k(\mathbf{a}\times\mathbf{b})$ (k = scalar)

[E$_4$] $\mathbf{a}\times\mathbf{a} = 0$

Note that the vector product is not only not commutative but also not associative; that is, in general $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. For as shown in Example 1.20, $\mathbf{g}_1\times(\mathbf{g}_1\times\mathbf{g}_2) = \mathbf{g}_1\times\mathbf{g}_3 = -\mathbf{g}_2$, whereas $(\mathbf{g}_1\times\mathbf{g}_1)\times\mathbf{g}_2 = 0\times\mathbf{g}_2 = 0$.

Example 1.21.

Consider the triangle ABC shown in Fig. 1-13. Let $\mathbf{a} = \mathbf{BC}$, $\mathbf{b} = \mathbf{AC}$, $\mathbf{c} = \mathbf{AB} = \mathbf{b}-\mathbf{a}$, $\alpha = \measuredangle(\mathbf{b}, \mathbf{c})$, $\beta = \measuredangle(\mathbf{c}, \mathbf{a})$, and $\gamma = \measuredangle(\mathbf{a}, \mathbf{b})$. Now,

$$0 = \mathbf{c}\times\mathbf{c} = \mathbf{c}\times(\mathbf{b}-\mathbf{a}) = \mathbf{c}\times\mathbf{b}-\mathbf{c}\times\mathbf{a}$$

or

$$\mathbf{c}\times\mathbf{b} = \mathbf{c}\times\mathbf{a}$$

Similarly,

$$\mathbf{c}\times\mathbf{b} = (\mathbf{b}-\mathbf{a})\times\mathbf{b} = \mathbf{b}\times\mathbf{b}-\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{a}$$

Hence

$$\mathbf{c}\times\mathbf{b} = \mathbf{c}\times\mathbf{a} = \mathbf{b}\times\mathbf{a}$$

But then

$$|\mathbf{c}\times\mathbf{b}| = |\mathbf{c}\times\mathbf{a}| = |\mathbf{b}\times\mathbf{a}|$$

or

$$|\mathbf{c}|\,|\mathbf{b}|\sin\alpha = |\mathbf{c}|\,|\mathbf{a}|\sin\beta = |\mathbf{b}|\,|\mathbf{a}|\sin\gamma$$

which gives the law of sines

$$\frac{\sin\alpha}{|\mathbf{a}|} = \frac{\sin\beta}{|\mathbf{b}|} = \frac{\sin\gamma}{|\mathbf{c}|}$$

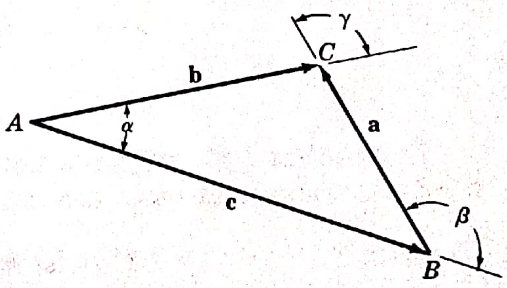


Fig. 1-13

TRIPLE PRODUCTS AND VECTOR IDENTITIES

The product $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$ is called the *mixed or triple scalar product*. Note that parentheses are not required; for this can only mean $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, the scalar product of the vector $\mathbf{a}$ and the vector $\mathbf{b}\times\mathbf{c}$. This product is also conveniently given in terms of a determinant. For, let

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3, \quad \mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3, \quad \mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$$

Then

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)\cdot\det\begin{pmatrix}\mathbf{e}_1 & b_1 & c_1\\ \mathbf{e}_2 & b_2 & c_2\\ \mathbf{e}_3 & b_3 & c_3\end{pmatrix}\\
&= a_1(b_2c_3 - b_3c_2) - a_2(c_3b_1 - c_1b_3) + a_3(b_1c_2 - b_2c_1) \qquad (1.8)\\
&= \det\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}
\end{aligned}$$

It follows from properties of the determinant that

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{c}\cdot\mathbf{a}\times\mathbf{b} = \mathbf{b}\cdot\mathbf{c}\times\mathbf{a} = -(\mathbf{b}\cdot\mathbf{a}\times\mathbf{c}) = -(\mathbf{c}\cdot\mathbf{b}\times\mathbf{a}) = -(\mathbf{a}\cdot\mathbf{c}\times\mathbf{b}) \qquad (1.9)$$

In particular, it follows that $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}$. Thus we can drop the dot and cross in the notation of the triple scalar product and use instead the notation

$$[\mathbf{abc}] = \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}$$

As immediate consequence of Theorem 1.3 and equation *(1.8)* we have

Theorem 1.7. $[\mathbf{abc}] = 0$ if and only if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly dependent.

A number of useful identities relate vector and scalar products of vectors. A basic identity, which is derived in Problem 1.35, is

Theorem 1.8. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

Others, easily derived from the above, are

$[\mathbf{F}_1]$ $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$

$[\mathbf{F}_2]$ $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d}$

Example 1.22.

Let $\mathbf{u} = \mathbf{c}\times\mathbf{d}$. Then

$$\mathbf{a}\times\mathbf{b}\cdot\mathbf{u} = \mathbf{a}\cdot\mathbf{b}\times\mathbf{u} = \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})] = \mathbf{a}\cdot[(\mathbf{b}\cdot\mathbf{d})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{d}]$$

where we used *(1.9)* and Theorem 1.8. It follows that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

which proves $[\mathbf{F}_1]$ above.

Solved Problems

VECTOR ADDITION

1.1. Prove properties $[\mathbf{A}_1]$ through $[\mathbf{A}_4]$ for vector addition. That is, prove that $[\mathbf{A}_1]$ $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$, $[\mathbf{A}_2]$ $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$, $[\mathbf{A}_3]$ $\mathbf{a}+\mathbf{0} = \mathbf{a}$, $[\mathbf{A}_4]$ $\mathbf{a}+(-\mathbf{a}) = \mathbf{0}$.

$$[\mathbf{A}_1]: \quad \mathbf{a}+\mathbf{b} = (a_1+b_1, a_2+b_2, a_3+b_3) = (b_1+a_1, b_2+a_2, b_3+a_3) = \mathbf{b}+\mathbf{a}$$

$$\begin{aligned}
[\mathbf{A}_2]: \quad (\mathbf{a}+\mathbf{b})+\mathbf{c} &= [(a_1+b_1)+c_1, (a_2+b_2)+c_2, (a_3+b_3)+c_3]\\
&= [a_1+(b_1+c_1), a_2+(b_2+c_2), a_3+(b_3+c_3)] = \mathbf{a}+(\mathbf{b}+\mathbf{c})
\end{aligned}$$

$$[\mathbf{A}_3]: \quad \mathbf{a}+\mathbf{0} = (a_1+0, a_2+0, a_3+0) = (a_1, a_2, a_3) = \mathbf{a}$$

$$[\mathbf{A}_4]: \quad \mathbf{a}+(-\mathbf{a}) = (a_1-a_1, a_2-a_2, a_3-a_3) = (0, 0, 0) = \mathbf{0}$$

1.2. In the parallelepiped shown in Fig. 1-14 let $\mathbf{a} = \mathbf{OP}$, $\mathbf{b} = \mathbf{OR}$, $\mathbf{c} = \mathbf{OS}$. Find $\mathbf{OV}$, $\mathbf{VQ}$ and $\mathbf{RT}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

$$\begin{aligned}\mathbf{OV} &= \mathbf{OR} + \mathbf{RV} = \mathbf{OR} + \mathbf{OS} = \mathbf{b} + \mathbf{c}\\ \mathbf{VQ} &= \mathbf{VR} + \mathbf{RQ} = -\mathbf{RV} + \mathbf{RQ}\\ &= -\mathbf{OS} + \mathbf{OP} = -\mathbf{c} + \mathbf{a}\\ \mathbf{RT} &= \mathbf{RS} + \mathbf{ST} = \mathbf{RO} + \mathbf{OS} + \mathbf{ST}\\ &= -\mathbf{b} + \mathbf{c} + \mathbf{a}\end{aligned}$$

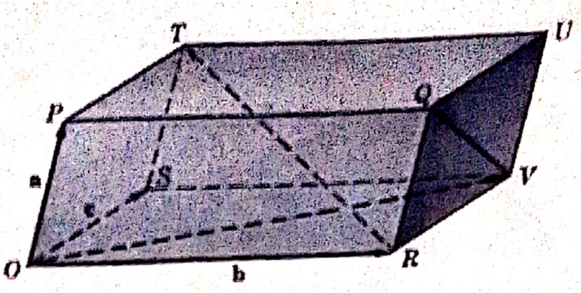


Fig. 1-14

1.3. It has been shown (Example 1.4.) using the properties $[A_1]$ through $[A_4]$ that the vector equation $\mathbf{a} + \mathbf{x} = \mathbf{b}$ has a *unique* solution $\mathbf{x} = \mathbf{b} + (-\mathbf{a}) = \mathbf{b} - \mathbf{a}$. Using this result, show that:

(a) the vector $\mathbf{0}$ is unique, that is, if $\mathbf{0}' + \mathbf{a} = \mathbf{a}$, then $\mathbf{0}' = \mathbf{0}$;

(b) the vector $-\mathbf{a}$ is unique, that is if $\mathbf{a}' + \mathbf{a} = \mathbf{0}$, then $\mathbf{a}' = -\mathbf{a}$;

(c) $-(-\mathbf{a}) = \mathbf{a}$ for all $\mathbf{a}$.

(a) follows from the uniqueness of the solution to the equation $\mathbf{x} + \mathbf{a} = \mathbf{a}$.

(b) follows from the uniqueness of the solution to the equation $\mathbf{x} + \mathbf{a} = \mathbf{0}$.

(c) follows when we consider the equation $-\mathbf{a} + \mathbf{x} = \mathbf{0}$. This has the solution $\mathbf{x} = \mathbf{0} - (-\mathbf{a}) = -(-\mathbf{a})$. But also $-\mathbf{a} + \mathbf{a} = \mathbf{0}$; thus $-(-\mathbf{a}) = \mathbf{a}$, by uniqueness of the solution to the equation.

MULTIPLICATION BY A SCALAR

1.4. Prove properties $[B_1]$ through $[B_3]$ for multiplication of a vector by a scalar. That is, prove that: $[B_1]$ $k_1(k_2\mathbf{a}) = (k_1k_2)\mathbf{a}$; $[B_2]$ $(k_1 + k_2)\mathbf{a} = k_1\mathbf{a} + k_2\mathbf{a}$, $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$; $[B_3]$ $1\mathbf{a} = \mathbf{a}$.

$$\begin{aligned}[B_1]:\quad k_1(k_2\mathbf{a}) &= (k_1(k_2a_1), k_1(k_2a_2), k_1(k_2a_3))\\ &= ((k_1k_2)a_1, (k_1k_2)a_2, (k_1k_2)a_3) = (k_1k_2)\mathbf{a}\end{aligned}$$

$$\begin{aligned}[B_2]:\quad (k_1 + k_2)\mathbf{a} &= ((k_1 + k_2)a_1, (k_1 + k_2)a_2, (k_1 + k_2)a_3)\\ &= (k_1a_1 + k_2a_1, k_1a_2 + k_2a_2, k_1a_3 + k_2a_3) = k_1\mathbf{a} + k_2\mathbf{a}\\ k(\mathbf{a} + \mathbf{b}) &= (k(a_1 + b_1), k(a_2 + b_2), k(a_3 + b_3))\\ &= (ka_1 + kb_1, ka_2 + kb_2, ka_3 + kb_3) = k\mathbf{a} + k\mathbf{b}\end{aligned}$$

$$[B_3]:\quad 1\mathbf{a} = (1a_1, 1a_2, 1a_3) = (a_1, a_2, a_3) = \mathbf{a}$$

1.5. If $\mathbf{a} = \mathbf{u}_1 - 2\mathbf{u}_2 + 3\mathbf{u}_3$, $\mathbf{b} = \mathbf{u}_2 - \mathbf{u}_3$ and $\mathbf{c} = \mathbf{u}_1 + 2\mathbf{u}_2$, find $2\mathbf{a} - 3(\mathbf{b} - \mathbf{c})$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

$$\begin{aligned}2\mathbf{a} - 3(\mathbf{b} - \mathbf{c}) = 2\mathbf{a} - 3\mathbf{b} + 3\mathbf{c} &= 2(\mathbf{u}_1 - 2\mathbf{u}_2 + 3\mathbf{u}_3) - 3(\mathbf{u}_2 - \mathbf{u}_3) + 3(\mathbf{u}_1 + 2\mathbf{u}_2)\\ &= 2\mathbf{u}_1 - 4\mathbf{u}_2 + 6\mathbf{u}_3 - 3\mathbf{u}_2 + 3\mathbf{u}_3 + 3\mathbf{u}_1 + 6\mathbf{u}_2 = 5\mathbf{u}_1 - \mathbf{u}_2 + 9\mathbf{u}_3\end{aligned}$$

1.6. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and has one-half its magnitude.

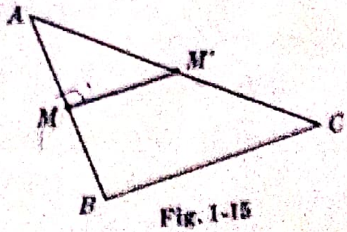


Fig. 1-15

Let M and M' be the midpoints of the sides AB and AC respectively of a triangle ABC shown in Fig. 1-15. Then $\mathbf{AM} = \frac{1}{2}\mathbf{AB}$, $\mathbf{AM'} = \frac{1}{2}\mathbf{AC}$ and $\mathbf{MM'} = \mathbf{AM'} - \mathbf{AM} = \frac{1}{2}(\mathbf{AC} - \mathbf{AB}) = \frac{1}{2}\mathbf{BC}$. Thus MM' is parallel to $\mathbf{BC}$ and has half the magnitude.

1.7. Let $\mathbf{a} = \mathbf{OA}$, $\mathbf{b} = \mathbf{OB}$, $\mathbf{b} \neq \mathbf{a}$ and $\mathbf{c} = \mathbf{OC}$ as shown in Fig. 1-16. Show that C lies on the line L determined by A and B if and only if $\mathbf{c} = k_1\mathbf{a} + k_2\mathbf{b}$ where $k_1 + k_2 = 1$.

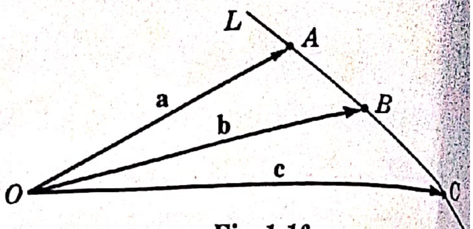


Fig. 1-16

If C is on L, then $\mathbf{BA} = \mathbf{a} - \mathbf{b}$ and $\mathbf{BC} = \mathbf{c} - \mathbf{b}$ are parallel. Hence there exists k such that

$$\mathbf{c} - \mathbf{b} = k(\mathbf{a} - \mathbf{b}) \quad \text{or} \quad \mathbf{c} = k\mathbf{a} + (1-k)\mathbf{b} = k_1\mathbf{a} + k_2\mathbf{b}$$

where $k_1 + k_2 = k + 1 - k = 1$. Conversely, if $\mathbf{c} = k_1\mathbf{a} + k_2\mathbf{b}$, where $k_1 + k_2 = 1$, $\mathbf{b} \neq \mathbf{a}$, then

$$\mathbf{c} - \mathbf{b} = k_1\mathbf{a} + k_2\mathbf{b} - \mathbf{b} = k_1\mathbf{a} - (1-k_2)\mathbf{b} = k_1\mathbf{a} - k_1\mathbf{b} = k_1(\mathbf{a} - \mathbf{b})$$

That is, $\mathbf{c} - \mathbf{b} = \mathbf{BC}$ and $\mathbf{a} - \mathbf{b} = \mathbf{BA}$ are parallel, so that C is on the line determined by A and

LINEAR DEPENDENCE AND INDEPENDENCE

1.8. Show that the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are dependent if and only if one of the vector is a linear combination of the others.

Suppose, say, $\mathbf{u}_1$ is a linear combination of $\mathbf{u}_2, \ldots, \mathbf{u}_n$; i.e. $\mathbf{u}_1 = k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n$. The $\mathbf{u}_1 - k_2\mathbf{u}_2 - \cdots - k_n\mathbf{u}_n = \mathbf{0}$ where at least the coefficient 1 of $\mathbf{u}_1$ is not zero. Hence $\mathbf{u}_1, \ldots, \mathbf{u}_n$ ar dependent.

Conversely, if $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are dependent, then there exists $k_1, \ldots, k_n$ not all zero such tha $k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n = \mathbf{0}$. Suppose, say, $k_1 \neq 0$; then $\mathbf{u}_1 = -(k_2/k_1)\mathbf{u}_2 - \cdots - (k_n/k_1)\mathbf{u}_n$ and s $\mathbf{u}_1$ is a linear combination of $\mathbf{u}_2, \ldots, \mathbf{u}_n$.

1.9. Prove that a set of vectors which contains a linearly dependent subset is linearly dependent.

Suppose, say, the subset $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ of the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is dependent. Then there exist $k_1, \ldots, k_k$ not all zero such that $k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_k\mathbf{u}_k = \mathbf{0}$. But then

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_k\mathbf{u}_k + 0\mathbf{u}_{k+1} + \cdots + 0\mathbf{u}_n = 0$$

and so $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are dependent.

1.10. Prove Theorem 1.1: If $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are independent and

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n = k_1'\mathbf{u}_1 + k_2'\mathbf{u}_2 + \cdots + k_n'\mathbf{u}_n$$

then $k_1 = k_1'$, $k_2 = k_2'$, $\ldots$, $k_n = k_n'$.

Suppose some $k_j \neq k_j'$; then

$$(k_1 - k_1')\mathbf{u}_1 + (k_2 - k_2')\mathbf{u}_2 + \cdots + (k_j - k_j')\mathbf{u}_j + \cdots + (k_n - k_n')\mathbf{u}_n = \mathbf{0}$$

where $k_j - k_j' \neq 0$. But this implies $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are dependent, which is a contradiction.

BASES AND COMPONENTS

1.11. Show that any vector in E^3 can be written as a linear combination of three independent vectors; hence three independent vectors form a basis in E^3.

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent, then the equation

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$$

has only the solution $x = y = z = 0$. Equivalently, the system

$$\begin{aligned} xa_1 + yb_1 + zc_1 &= 0 \\ xa_2 + yb_2 + zc_2 &= 0 \\ xa_3 + yb_3 + zc_3 &= 0 \end{aligned}$$

has only the trivial solution $x = y = z = 0$, which is the case only if the matrix of coefficients has determinant not zero, that is,

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0$$

But then for any vector $\mathbf{u} = (u_1, u_2, u_3)$ in E^3 the system

$$\begin{aligned} xa_1 + yb_1 + zc_1 &= u_1 \\ xa_2 + yb_2 + zc_2 &= u_2 \\ xa_3 + yb_3 + zc_3 &= u_3 \end{aligned}$$

has a solution $x = k_1$, $y = k_2$, $z = k_3$ which means that $\mathbf{u} = k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c}$ as required.

1.12. Show that any four or more vectors in E^3 are linearly dependent.

Consider the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \ldots, \mathbf{u}_n$. We can assume that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, are independent. For otherwise, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \ldots, \mathbf{u}_n$, having a dependent subset, would be dependent. But if $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are independent, they form a basis and so $\mathbf{u}_4 = k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3$, which implies $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are dependent. Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \ldots, \mathbf{u}_n$ are dependent.

1.13. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and let $\mathbf{a} = \mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3$, $\mathbf{b} = \mathbf{u}_2 - \mathbf{u}_3$ and $\mathbf{c} = -\mathbf{u}_2$. Find the components of $2\mathbf{a} - \mathbf{b} - 2\mathbf{c}$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

$$\begin{aligned} 2\mathbf{a} - \mathbf{b} - 2\mathbf{c} &= 2(\mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3) - (\mathbf{u}_2 - \mathbf{u}_3) - 2(-\mathbf{u}_2) \\ &= 2\mathbf{u}_1 - 2\mathbf{u}_2 + 4\mathbf{u}_3 - \mathbf{u}_2 + \mathbf{u}_3 + 2\mathbf{u}_2 = 2\mathbf{u}_1 - \mathbf{u}_2 + 5\mathbf{u}_3 \end{aligned}$$

Thus the components of $2\mathbf{a} - \mathbf{b} - 2\mathbf{c}$ with respect to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are $2, -1, 5$.

SCALAR PRODUCT

1.14. Prove properties $[\mathbf{C}_1]$ through $[\mathbf{C}_4]$, page 5, for the scalar product of vectors.

$[\mathbf{C}_1]$: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \mathbf{b} \cdot \mathbf{a}$

$[\mathbf{C}_2]$: $(k\mathbf{a}) \cdot \mathbf{b} = ka_1b_1 + ka_2b_2 + ka_3b_3 = k(a_1b_1 + a_2b_2 + a_3b_3) = k(\mathbf{a} \cdot \mathbf{b})$

$[\mathbf{C}_3]$:
$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3 = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \end{aligned}$$

$[\mathbf{C}_4]$: Clearly $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 \geq 0$ and $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = 0$ iff $a_1 = a_2 = a_3 = 0$.

1.15. In the triangle OAB shown in Fig. 1-17, let $\mathbf{a} = \mathbf{OA}$ and $\mathbf{b} = \mathbf{OB}$. If $|\mathbf{OA}| = 2$, $|\mathbf{OB}| = 3$ and $\measuredangle AOB = 30°$, find (*a*) $\mathbf{a} \cdot \mathbf{b}$, (*b*) $P_{\mathbf{a}}(\mathbf{b})$, (*c*) $\mathbf{P}_{\mathbf{a}}(\mathbf{b})$.

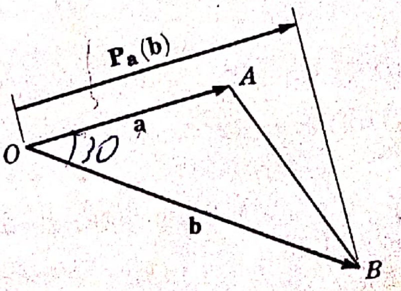


Fig. 1-17

(*a*)
$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}| \cos \measuredangle(\mathbf{a}, \mathbf{b}) \\ &= (2)(3) \cos 30° = 3\sqrt{3} \end{aligned}$$

(*b*) $P_{\mathbf{a}}(\mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{a}| = 3\sqrt{3}/2$

1.16. Prove the *Cauchy-Schwarz* inequality, $|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}|\,|\mathbf{b}|$, with equality holding if an only if $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent.

The inequality is clearly valid if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, and so we may assume that $\mathbf{a}, \mathbf{b} \ne$ Then from $[\mathbf{C}_4]$,

$$0 \le \left(\sqrt{\frac{|\mathbf{b}|}{|\mathbf{a}|}}\,\mathbf{a} \pm \sqrt{\frac{|\mathbf{a}|}{|\mathbf{b}|}}\,\mathbf{b}\right)\cdot\left(\sqrt{\frac{|\mathbf{b}|}{|\mathbf{a}|}}\,\mathbf{a} \pm \sqrt{\frac{|\mathbf{a}|}{|\mathbf{b}|}}\,\mathbf{b}\right) \le 2|\mathbf{a}|\,|\mathbf{b}| \pm 2\mathbf{a}\cdot\mathbf{b}$$

or $\pm 2\mathbf{a}\cdot\mathbf{b} \le 2|\mathbf{a}|\,|\mathbf{b}|$ which gives the desired inequality $|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}|\,|\mathbf{b}|$. The remaining statemen follows from the fact that equality can hold if and only if either $\sqrt{\frac{|\mathbf{b}|}{|\mathbf{a}|}}\mathbf{a} + \sqrt{\frac{|\mathbf{a}|}{|\mathbf{b}|}}\mathbf{b} = \mathbf{0}$ o $\sqrt{\frac{|\mathbf{b}|}{|\mathbf{a}|}}\mathbf{a} - \sqrt{\frac{|\mathbf{a}|}{|\mathbf{b}|}}\mathbf{b} = \mathbf{0}$ which is the case if and only if $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent.

1.17. Prove the *triangle inequality*, $|\mathbf{a}\pm\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$.

$$|\mathbf{a}\pm\mathbf{b}|^2 = (\mathbf{a}\pm\mathbf{b})\cdot(\mathbf{a}\pm\mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 \pm 2(\mathbf{a}\cdot\mathbf{b}) \le |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}|\,|\mathbf{b}| \le (|\mathbf{a}| + |\mathbf{b}|)^2$$

and the desired result follows upon taking square roots.

1.18. Show that $||\mathbf{a}| - |\mathbf{b}|| \le |\mathbf{a}\pm\mathbf{b}|$ for all $\mathbf{a}$ and $\mathbf{b}$.

From the triangle inequality,

$$|\mathbf{a}| = |\mathbf{a}\pm\mathbf{b}\mp\mathbf{b}| \le |\mathbf{a}\pm\mathbf{b}| + |\mathbf{b}| \quad\text{or}\quad |\mathbf{a}| - |\mathbf{b}| \le |\mathbf{a}\pm\mathbf{b}|$$

Also $\quad |\mathbf{b}| = |\pm\mathbf{b}| = |\mathbf{a}\pm\mathbf{b}-\mathbf{a}| \le |\mathbf{a}\pm\mathbf{b}| + |\mathbf{a}| \quad$ or $\quad |\mathbf{b}| - |\mathbf{a}| \le |\mathbf{a}\pm\mathbf{b}|$

Thus $\; ||\mathbf{a}| - |\mathbf{b}|| = \operatorname{Max}(|\mathbf{a}| - |\mathbf{b}|,\, |\mathbf{b}| - |\mathbf{a}|) \le |\mathbf{a}\pm\mathbf{b}|$, which is the required result.

ORTHOGONAL VECTORS

1.19. Let $\mathbf{c}$ be orthogonal to $\mathbf{a}$ and $\mathbf{b}$. Show that $\mathbf{c}$ is orthogonal to $k_1\mathbf{a} + k_2\mathbf{b}$ for all k_1, k_2.

Since $\mathbf{c}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{c}\cdot\mathbf{a} = 0$ and $\mathbf{c}\cdot\mathbf{b} = 0$. Hence

$$\mathbf{c}\cdot(k_1\mathbf{a} + k_2\mathbf{b}) = k_1(\mathbf{c}\cdot\mathbf{a}) + k_2(\mathbf{c}\cdot\mathbf{b}) = 0$$

Thus $\mathbf{c}$ is orthogonal to $k_1\mathbf{a} + k_2\mathbf{b}$.

1.20. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and define

$$\mathbf{a} = \mathbf{u}_1,\quad \mathbf{b} = \mathbf{u}_2 - \mathbf{P}_{\mathbf{a}}(\mathbf{u}_2),\quad \mathbf{c} = \mathbf{u}_3 - \mathbf{P}_{\mathbf{a}}(\mathbf{u}_3) - \mathbf{P}_{\mathbf{b}}(\mathbf{u}_3)$$

Show that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are nonzero mutually orthogonal vectors.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot(\mathbf{u}_2 - \mathbf{P}_{\mathbf{a}}(\mathbf{u}_2)) = \mathbf{a}\cdot[\mathbf{u}_2 - (\mathbf{a}\cdot\mathbf{u}_2)\mathbf{a}/|\mathbf{a}|^2]\\ &= \mathbf{a}\cdot\mathbf{u}_2 - (\mathbf{a}\cdot\mathbf{u}_2)(\mathbf{a}\cdot\mathbf{a})/|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{u}_2 - \mathbf{a}\cdot\mathbf{u}_2 = 0\end{aligned}$$

and so $\mathbf{a}\perp\mathbf{b}$. Also,

$$\begin{aligned}\mathbf{a}\cdot\mathbf{c} &= \mathbf{a}\cdot[\mathbf{u}_3 - \mathbf{P}_{\mathbf{a}}(\mathbf{u}_3) - \mathbf{P}_{\mathbf{b}}(\mathbf{u}_3)] = \mathbf{a}\cdot[\mathbf{u}_3 - (\mathbf{a}\cdot\mathbf{u}_3)\mathbf{a}/|\mathbf{a}|^2 - (\mathbf{b}\cdot\mathbf{u}_3)\mathbf{b}/|\mathbf{b}|^2]\\ &= \mathbf{a}\cdot\mathbf{u}_3 - \mathbf{a}\cdot\mathbf{u}_3 - (\mathbf{b}\cdot\mathbf{u}_3)(\mathbf{a}\cdot\mathbf{b})/|\mathbf{b}|^2\end{aligned}$$

Since $\mathbf{a}\cdot\mathbf{b} = 0$, $\mathbf{a}\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{u}_3 - \mathbf{a}\cdot\mathbf{u}_3 = 0$ and so $\mathbf{a}\perp\mathbf{c}$. Finally,

$$\begin{aligned}\mathbf{b}\cdot\mathbf{c} &= \mathbf{b}\cdot[\mathbf{u}_3 - (\mathbf{a}\cdot\mathbf{u}_3)\mathbf{a}/|\mathbf{a}|^2 - (\mathbf{b}\cdot\mathbf{u}_3)\mathbf{b}/|\mathbf{b}|^2]\\ &= (\mathbf{b}\cdot\mathbf{u}_3) - (\mathbf{a}\cdot\mathbf{u}_3)(\mathbf{a}\cdot\mathbf{b})/|\mathbf{a}|^2 - (\mathbf{b}\cdot\mathbf{u}_3)(\mathbf{b}\cdot\mathbf{b})/|\mathbf{b}|^2 = (\mathbf{b}\cdot\mathbf{u}_3) - (\mathbf{b}\cdot\mathbf{u}_3) = 0\end{aligned}$$

Thus $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually orthogonal. They are also nonzero vectors, because: $\mathbf{a} = \mathbf{u}_1 \ne \mathbf{0}$; if $\mathbf{b} = \mathbf{0}$,

$$\mathbf{0} = \mathbf{b} = \mathbf{u}_2 - \mathbf{P}_{\mathbf{a}}(\mathbf{u}_2) = \mathbf{u}_2 - k\mathbf{a} = \mathbf{u}_2 - k\mathbf{u}_1$$

which is impossible since $\mathbf{u}_1$ and $\mathbf{u}_2$ are independent; if $\mathbf{c} = \mathbf{0}$,

$$\mathbf{0} = \mathbf{c} = \mathbf{u}_3 - \mathbf{P}_{\mathbf{a}}(\mathbf{u}_3) - \mathbf{P}_{\mathbf{b}}(\mathbf{u}_3) = \mathbf{u}_3 - k_1\mathbf{a} - k_2\mathbf{b} = \mathbf{u}_3 - k_1\mathbf{u}_1 - k_2(\mathbf{u}_2 - k\mathbf{u}_1) = \mathbf{u}_3 - k_3\mathbf{u}_1 - k_2\mathbf{u}_2$$

which is impossible since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are independent.

ORTHONORMAL BASES

1.21. Show that (*a*) $2\delta_{21} + 3\delta_{22} + 4\delta_{23} = 3$, (*b*) $\sum_{j=1}^{3} \delta_{ij}\mathbf{b}_j = \mathbf{b}_i$.

(*a*) $\delta_{ij} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$. Hence $2\delta_{21} + 3\delta_{22} + 4\delta_{23} = 2(0) + 3(1) + 4(0) = 3$.

(*b*) As above, the only contribution comes from δ_{ii}. Thus $\sum_{j=1}^{3} \delta_{ij}\mathbf{b}_j = \delta_{ii}\mathbf{b}_i = \mathbf{b}_i$.

1.22. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and let $\mathbf{v}_j = \sum_{i=1}^{3} a_{ij}\mathbf{u}_i$ and $\mathbf{u}_j = \sum_{i=1}^{3} b_{ij}\mathbf{v}_i$. Show that $\sum_{k=1}^{3} a_{ik}b_{kj} = \delta_{ij}$.

We write $\mathbf{u}_j = \sum_{i=1}^{3} \delta_{ij}\mathbf{u}_i = \sum_{k=1}^{3} b_{kj}\mathbf{v}_k$ where we have changed the name of the index from i to k. Also, $\mathbf{v}_k = \sum_{i=1}^{3} a_{ik}\mathbf{u}_i$. Substituting,

$$\sum_{i=1}^{3} \delta_{ij}\mathbf{u}_i = \sum_{k=1}^{3} b_{kj} \sum_{i=1}^{3} a_{ik}\mathbf{u}_i = \sum_{i=1}^{3}\left[\sum_{k=1}^{3} a_{ik}b_{kj}\right]\mathbf{u}_i$$

Since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are independent equate components and obtain $\delta_{ij} = \sum_{k=1}^{3} a_{ik}b_{kj}$.

1.23. Prove Theorem 1.4: If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is an orthonormal basis and $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$, then
(*a*) $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$, (*b*) $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, (*c*) $a_i = \mathbf{a}\cdot\mathbf{e}_i$, $i = 1, 2, 3$.

(*a*)
$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)\cdot(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \\ &= a_1b_1(\mathbf{e}_1\cdot\mathbf{e}_1) + a_1b_2(\mathbf{e}_1\cdot\mathbf{e}_2) + a_1b_3(\mathbf{e}_1\cdot\mathbf{e}_3) \\ &\quad + a_2b_1(\mathbf{e}_2\cdot\mathbf{e}_1) + a_2b_2(\mathbf{e}_2\cdot\mathbf{e}_2) + a_2b_3(\mathbf{e}_2\cdot\mathbf{e}_3) \\ &\quad + a_3b_1(\mathbf{e}_3\cdot\mathbf{e}_1) + a_3b_2(\mathbf{e}_3\cdot\mathbf{e}_2) + a_3b_3(\mathbf{e}_3\cdot\mathbf{e}_3) \\ &= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

In short, we have

$$\mathbf{a}\cdot\mathbf{b} = \left(\sum_{i=1}^{3} a_i\mathbf{e}_i\right)\cdot\left(\sum_{j=1}^{3} b_j\mathbf{e}_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_ib_j(\mathbf{e}_i\cdot\mathbf{e}_j)$$

or

$$\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_ib_j\delta_{ij} = \sum_{i=1}^{3} a_ib_i$$

(*b*) $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$

(*c*) $\mathbf{a}\cdot\mathbf{e}_i = \left(\sum_j a_j\mathbf{e}_j\right)\cdot\mathbf{e}_i = \sum_j a_j(\mathbf{e}_j\cdot\mathbf{e}_i) = \sum_j a_j\delta_{ji} = a_i$

1.24. Let $\mathbf{a} = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_2 - \mathbf{e}_3$. Find (*a*) $\mathbf{a}\cdot\mathbf{b}$, (*b*) $|\mathbf{a}|$, (*c*) $\mathbf{u}_\mathbf{a}$, (*d*) $P_\mathbf{a}(\mathbf{b})$, (*e*) $\mathbf{P}_\mathbf{a}(\mathbf{b})$, (*f*) $\cos\measuredangle(\mathbf{a},\mathbf{b})$, (*g*) $\mathbf{a}\cdot\mathbf{e}_1$, $\mathbf{a}\cdot\mathbf{e}_2$, $\mathbf{a}\cdot\mathbf{e}_3$, (*h*) direction cosines of $\mathbf{a}$.

(*a*) $\mathbf{a}\cdot\mathbf{b} = (1)(0) + (-2)(1) + (3)(-1) = -5$

(*b*) $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{14}$

(*c*) $\mathbf{u}_\mathbf{a} = \mathbf{a}/|\mathbf{a}| = (1/\sqrt{14})(\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3)$

(*d*) $P_\mathbf{a}(\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})/|\mathbf{a}| = -5/\sqrt{14}$

(*e*) $\mathbf{P}_\mathbf{a}(\mathbf{b}) = P_\mathbf{a}(\mathbf{b})\mathbf{u}_\mathbf{a} = -(5/14)(\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3)$

(*f*) $\cos\measuredangle(\mathbf{a},\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})/|\mathbf{a}|\,|\mathbf{b}| = (-5/\sqrt{14}\sqrt{2}) = -5/(2\sqrt{7})$

(*g*) $\mathbf{a}\cdot\mathbf{e}_1 = 1$, $\mathbf{a}\cdot\mathbf{e}_2 = -2$, $\mathbf{a}\cdot\mathbf{e}_3 = 3$

(*h*) $\cos\measuredangle(\mathbf{a},\mathbf{e}_1) = a_1/|\mathbf{a}| = 1/\sqrt{14}$, $\cos\measuredangle(\mathbf{a},\mathbf{e}_2) = a_2/|\mathbf{a}| = -2/\sqrt{14}$, $\cos\measuredangle(\mathbf{a},\mathbf{e}_3) = a_3/|\mathbf{a}| = 3/\sqrt{14}$

1.25. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be an arbitrary basis. Show that there exists a unique basis $\mathbf{v}_1,$ such that

$$\mathbf{v}_1\cdot\mathbf{u}_1 = 1 \qquad \mathbf{v}_2\cdot\mathbf{u}_1 = 0 \qquad \mathbf{v}_3\cdot\mathbf{u}_1 = 0$$
$$\mathbf{v}_1\cdot\mathbf{u}_2 = 0 \qquad \mathbf{v}_2\cdot\mathbf{u}_2 = 1 \qquad \mathbf{v}_3\cdot\mathbf{u}_2 = 0$$
$$\mathbf{v}_1\cdot\mathbf{u}_3 = 0 \qquad \mathbf{v}_2\cdot\mathbf{u}_3 = 0 \qquad \mathbf{v}_3\cdot\mathbf{u}_3 = 1$$

or $\mathbf{v}_i\cdot\mathbf{u}_j = \delta_{ij},\ i,j = 1,2,3.$ The basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is called the *dual or reciprocal* to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. Accordingly if $\mathbf{a} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$ and $\mathbf{b} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3$,

$$\mathbf{a}\cdot\mathbf{b} = \left(\sum_i a_i\mathbf{u}_i\right)\cdot\left(\sum_j b_j\mathbf{v}_j\right) = \sum_i\sum_j a_ib_j(\mathbf{u}_i\cdot\mathbf{v}_j) = \sum_i\sum_j a_ib_j\delta_{ij} = \sum_i a_ib_i$$

Observe that an orthonormal basis is its own dual.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis and let

$$\begin{aligned}\mathbf{u}_1 &= a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 + a_{13}\mathbf{e}_3\\ \mathbf{u}_2 &= a_{21}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{23}\mathbf{e}_3\\ \mathbf{u}_3 &= a_{31}\mathbf{e}_1 + a_{32}\mathbf{e}_2 + a_{33}\mathbf{e}_3\end{aligned}$$

and

$$\mathbf{v}_1 = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

Then

$$\begin{aligned}\mathbf{v}_1\cdot\mathbf{u}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 1\\ \mathbf{v}_1\cdot\mathbf{u}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0\\ \mathbf{v}_1\cdot\mathbf{u}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0\end{aligned}$$

is a system of three equations for x_1, x_2, x_3. Since $\det(a_{ij}) \neq 0$, there exists a unique solut $\mathbf{v}_1 = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$. Similarly we have unique solutions for $\mathbf{v}_2$ and $\mathbf{v}_3$. It remains to sh that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent and hence form a basis. We consider

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \sum_{i=1}^{3} k_i\mathbf{v}_i = 0$$

If we multiply by $\mathbf{u}_j$, $j = 1,2,3,$ we obtain

$$\left[\sum_{i=1}^{3} k_i\mathbf{v}_i\right]\cdot\mathbf{u}_j = \sum_{i=1}^{3} k_i(\mathbf{v}_i\cdot\mathbf{u}_j) = \sum_{i=1}^{3} k_i\delta_{ij} = k_j = 0, \qquad j = 1,2,3$$

Thus $k_1 = k_2 = k_3 = 0$ and so $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent and form a basis.

ORIENTATION

1.26. Show that the triplet $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, where $\mathbf{v}_1 = 2\mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3$, $\mathbf{v}_2 = \mathbf{u}_2 + \mathbf{u}_3$ ar $\mathbf{v}_3 = -\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3$, has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

The determinant of the components is $\det\begin{pmatrix} 2 & 0 & -1\\ -1 & 1 & 2\\ 2 & 1 & 1\end{pmatrix} = 1 > 0.$

Hence $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

1.27. Show that orientation is an *equivalence relation* on the set of all ordered bases in E That is, show that:

(*a*) $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ for all $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(*b*) If $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, then $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ has the sam orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(*c*) If $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ has the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the sam orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, then $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$

(*a*) We write $\mathbf{v}_j = \sum_{i=1}^{3} \delta_{ij}\mathbf{v}_i$, $j = 1, 2, 3$. Since $\det(\delta_{ij}) = 1$, $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(*b*) Let $\mathbf{v}_j = \sum_i a_{ij}\mathbf{u}_i$ and $\mathbf{u}_j = \sum_i b_{ij}\mathbf{v}_i$. From Problem 1.22, $\sum_{k=1}^{3} a_{ik}b_{kj} = \delta_{ij}$. It is also easily verified by expanding that $\det\left(\sum_{k=1}^{3} a_{ik}b_{kj}\right) = \det(a_{ij})\det(b_{ij})$, and so

$$\det(b_{ij}) \quad = \quad \frac{\det(\delta_{ij})}{\det(a_{ij})} \quad = \quad \frac{1}{\det(a_{ij})}$$

Since $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, $\det(a_{ij}) > 0$. Hence $\det(b_{ij}) > 0$, and thus $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ has the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(*c*) Let $\mathbf{v}_k = \sum_{i=1}^{3} a_{ik}\mathbf{u}_i$ and $\mathbf{w}_j = \sum_{k=1}^{3} b_{kj}\mathbf{v}_k$. Substituting, we obtain

$$\mathbf{w}_j \quad = \quad \sum_{k=1}^{3} b_{kj} \sum_{i=1}^{3} a_{ik}\mathbf{u}_i \quad = \quad \sum_{i=1}^{3}\left(\sum_{k=1}^{3} a_{ik}b_{kj}\right)\mathbf{u}_i$$

Thus $\mathbf{w}_j = \sum_{i=1}^{3} c_{ij}\mathbf{u}_i$ where $c_{ij} = \sum_{k=1}^{3} a_{ik}b_{kj}$, $i, j = 1, 2, 3$. Also,

$$\det(c_{ij}) \quad = \quad \det\left(\sum a_{ik}b_{kj}\right) \quad = \quad \det(a_{ij})\det(b_{ij})$$

Since $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ has the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, then $\det(b_{ij}) > 0$ and $\det(a_{ij}) > 0$; hence $\det(c_{ij}) > 0$. Thus $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

VECTOR PRODUCT

1.28. Let $\mathbf{a} = 2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{c} = \mathbf{e}_2 + 2\mathbf{e}_3$. Determine (*a*) $\mathbf{a} \times \mathbf{b}$, (*b*) $\mathbf{b} \times \mathbf{a}$, (*c*) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, (*d*) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, (*e*) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, (*f*) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) - \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$.

$$(a)\quad \mathbf{a} \times \mathbf{b} \quad = \quad \det\begin{pmatrix} \mathbf{e}_1 & 2 & 1 \\ \mathbf{e}_2 & -1 & 2 \\ \mathbf{e}_3 & 1 & -1 \end{pmatrix} \quad = \quad \mathbf{e}_1 \det\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} - \mathbf{e}_2 \det\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} + \mathbf{e}_3 \det\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \quad -\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3$$

$$(b)\quad \mathbf{b} \times \mathbf{a} \quad = \quad \det\begin{pmatrix} \mathbf{e}_1 & 1 & 2 \\ \mathbf{e}_2 & 2 & -1 \\ \mathbf{e}_3 & -1 & 1 \end{pmatrix} \quad = \quad \mathbf{e}_1 - 3\mathbf{e}_2 - 5\mathbf{e}_3.$$

Observe that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

$$(c)\quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \quad = \quad (2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3) \times \det\begin{pmatrix} \mathbf{e}_1 & 1 & 0 \\ \mathbf{e}_2 & 2 & 1 \\ \mathbf{e}_3 & -1 & 2 \end{pmatrix}$$

$$= \quad (2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3) \times (5\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3) \quad = \quad \det\begin{pmatrix} \mathbf{e}_1 & 2 & 5 \\ \mathbf{e}_2 & -1 & -2 \\ \mathbf{e}_3 & 1 & 1 \end{pmatrix} \quad = \quad \mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3$$

$$(d)\quad (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \quad = \quad (-\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3) \times (\mathbf{e}_2 + 2\mathbf{e}_3) \quad = \quad \det\begin{pmatrix} \mathbf{e}_1 & -1 & 0 \\ \mathbf{e}_2 & 3 & 1 \\ \mathbf{e}_3 & 5 & 2 \end{pmatrix} \quad = \quad \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$$

Observe that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

$$(e)\quad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \quad = \quad (-\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3) \cdot (\mathbf{e}_2 + 2\mathbf{e}_3) \quad = \quad (-1)(0) + (3)(1) + (5)(2) \quad = \quad 13$$

(*f*) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \det\begin{pmatrix} \mathbf{e}_1 & 2 & 1 \\ \mathbf{e}_2 & -1 & 3 \\ \mathbf{e}_3 & 1 & 1 \end{pmatrix} = -4\mathbf{e}_1 - \mathbf{e}_2 + 7\mathbf{e}_3$, $\mathbf{a}\times\mathbf{b} = -\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3$, and $\mathbf{a}\times\mathbf{c} = -3\mathbf{e}_1 -$ $4\mathbf{e}_2 + 2\mathbf{e}_3$. Then

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) - \mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c} = (-4\mathbf{e}_1 - \mathbf{e}_2 + 7\mathbf{e}_3) - (-\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3) - (-3\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_3) = 0$$

1.29. Prove that (*a*) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$, (*b*) $(k\mathbf{a})\times\mathbf{b} = k(\mathbf{a}\times\mathbf{b})$.

Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$, $\mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$.

(*a*)
$$\begin{aligned}\mathbf{a}\times(\mathbf{b}+\mathbf{c}) &= [a_2(b_3+c_3) - a_3(b_2+c_2)]\mathbf{e}_1 + [a_3(b_1+c_1) - a_1(b_3+c_3)]\mathbf{e}_2 \\ &\quad + [a_1(b_2+c_2) - a_2(b_1+c_1)]\mathbf{e}_3 \\ &= (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3 \\ &\quad + (a_2c_3 - a_3c_2)\mathbf{e}_1 + (a_3c_1 - a_1c_3)\mathbf{e}_2 + (a_1c_2 - a_2c_1)\mathbf{e}_3 \\ &= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}\end{aligned}$$

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(*b*)
$$\begin{aligned}(k\mathbf{a})\times\mathbf{b} &= (ka_2b_3 - ka_3b_2)\mathbf{e}_1 + (ka_3b_1 - ka_1b_3)\mathbf{e}_2 + (ka_1b_2 - ka_2b_1)\mathbf{e}_3 \\ &= k[(a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3] = k(\mathbf{a}\times\mathbf{b})\end{aligned}$$

1.30. Show that $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 - |\mathbf{a}\cdot\mathbf{b}|^2$.

Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$.

$$\begin{aligned}|\mathbf{a}\times\mathbf{b}|^2 = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) &= [(a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3] \\ &\qquad \cdot [(a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3] \\ &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2 \\ &\qquad - 2a_2b_2a_3b_3 - 2a_1b_1a_3b_3 - 2a_1b_1a_2b_2\end{aligned}$$

$$\begin{aligned}|\mathbf{a}|^2\,|\mathbf{b}|^2 - |\mathbf{a}\cdot\mathbf{b}|^2 &= (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{b}) \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 - 2a_1b_1a_2b_2 - 2a_1b_1a_3b_3 - 2a_2b_2a_3b_3\end{aligned}$$

The required identity follows by comparing the above.

1.31. Prove Theorem 1.5:

(i) $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$, $\theta = \measuredangle(\mathbf{a},\mathbf{b})$

(ii) a. $(\mathbf{a}\times\mathbf{b}) \perp \mathbf{a}$ and $(\mathbf{a}\times\mathbf{b}) \perp \mathbf{b}$

b. If $(\mathbf{a}\times\mathbf{b}) \neq 0$, then $(\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b})$ has the same orientation as $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

(i) Using the preceding problem,

$$\begin{aligned}|\mathbf{a}\times\mathbf{b}|^2 &= |\mathbf{a}|^2\,|\mathbf{b}|^2 - |\mathbf{a}\cdot\mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 - |\mathbf{a}|^2\,|\mathbf{b}|^2\cos^2\theta \\ &= |\mathbf{a}|^2\,|\mathbf{b}|^2(1 - \cos^2\theta) = |\mathbf{a}|^2\,|\mathbf{b}|^2\sin^2\theta = (|\mathbf{a}|\,|\mathbf{b}|\sin\theta)^2\end{aligned}$$

Since $\sin\theta \geqq 0$ for $0 \leqq \theta \leqq \pi$, we have $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$.

(ii) a. Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$.

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} &= [(a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3]\cdot(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \\ &= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1 = 0\end{aligned}$$

Similarly, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0$. Hence $(\mathbf{a}\times\mathbf{b}) \perp \mathbf{a}$ and $(\mathbf{a}\times\mathbf{b}) \perp \mathbf{b}$.

b. The determinant of the components of $(\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b})$ is

$$\det\begin{pmatrix} a_1 & b_1 & (a_2b_3 - a_3b_2) \\ a_2 & b_2 & (a_3b_1 - a_1b_3) \\ a_3 & b_3 & (a_1b_2 - a_2b_1) \end{pmatrix} = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 = |\mathbf{a}\times\mathbf{b}|^2$$

1.32. Prove that the definition of the vector product is independent of the basis.

Let $\mathbf{c}$ and $\mathbf{c}'$ be the vector products of $\mathbf{a}$ and $\mathbf{b}$ with respect to two different right-handed orthonormal bases. We may assume that $\mathbf{a}$ and $\mathbf{b}$ are independent. Otherwise, from Theorem 1.6, $\mathbf{c} = \mathbf{c}' = \mathbf{0}$. From Theorem 1.5(ii), $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a basis and we can write $\mathbf{c}' = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. Also from Theorem 1.5(ii), $\mathbf{a} \cdot \mathbf{c}' = \alpha|\mathbf{a}|^2 + \beta(\mathbf{a} \cdot \mathbf{b}) = 0$ and $\mathbf{b} \cdot \mathbf{c}' = \alpha(\mathbf{b} \cdot \mathbf{a}) + \beta|\mathbf{b}|^2 = 0$. Since $\mathbf{a}$ and $\mathbf{b}$ are independent, $|\mathbf{a}|^2|\mathbf{b}|^2 - |a \cdot b|^2 \neq 0$. Hence $\alpha = \beta = 0$ and $\mathbf{c}' = \gamma\mathbf{c}$. Since $(\mathbf{a}, \mathbf{b}, \mathbf{c}')$ and $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ are both right-handed, $\gamma > 0$. From Theorem 1.5(i), $|\mathbf{c}'| = |\mathbf{c}| = \gamma|\mathbf{c}'|$. Thus $\gamma = 1$ and $\mathbf{c} = \mathbf{c}'$.

TRIPLE PRODUCTS

1.33. Let $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{b} = -\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{c} = -\mathbf{e}_2 + 2\mathbf{e}_3$. Find $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$.

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \;=\; \det\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix} \;=\; 5$$

1.34. Show that $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.

Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$, $\mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$. Then

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \;=\; \det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \;=\; -\det\begin{pmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{pmatrix} \;=\; \det\begin{pmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{pmatrix} \;=\; \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} \;=\; \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$$

1.35. Prove Theorem 1.8: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$, $\mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$. Then

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \times [(b_2c_3 - b_3c_2)\mathbf{e}_1 + (b_3c_1 - c_3b_1)\mathbf{e}_2 + (b_1c_2 - b_2c_1)\mathbf{e}_3] \\ &= (a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3)\mathbf{e}_1 \\ &\quad + (a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1)\mathbf{e}_2 \\ &\quad + (a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2)\mathbf{e}_3 \end{aligned}$$

Thus comparing with the above,

$$\begin{aligned} (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} &= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \\ &\quad - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3) \\ &= (a_2b_1c_2 + a_3b_1c_3 - a_2c_1b_2 - a_3c_1b_3)\mathbf{e}_1 \\ &\quad + (b_2a_1c_1 + b_2a_3c_3 - c_2a_1b_1 - c_2a_3b_3)\mathbf{e}_2 \\ &\quad + (b_3a_1c_1 + b_3a_2c_2 - c_3a_1b_1 - c_3a_2b_2)\mathbf{e}_3 \\ &= \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \end{aligned}$$

Supplementary Problems

1.36. In the tetrahedron $OPQR$ shown in Fig. 1-18, let $\mathbf{a} = \mathbf{OP}$, $\mathbf{b} = \mathbf{OQ}$, $\mathbf{c} = \mathbf{OR}$ and let M be the midpoint of edge RQ. Find $\mathbf{PM}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. *Ans.* $\mathbf{PM} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} - \mathbf{a}$

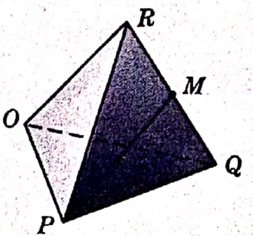

Fig. 1-18

1.37. Let $\mathbf{a} = 2\mathbf{u}_1 + \mathbf{u}_2 - 3\mathbf{u}_3$, $\mathbf{b} = \mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3$, $\mathbf{c} = -\mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3$. Find $3\mathbf{a} - 2\mathbf{b} + \mathbf{c}$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. *Ans.* $3\mathbf{u}_1 + 9\mathbf{u}_2 - 12\mathbf{u}_3$

1.38. Show that $|\mathbf{a} \pm \mathbf{b} \pm \mathbf{c}| \leq |\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}|$.

1.39. Show that the midpoints of the lines joining the midpoints of opposite sides of a quadrilateral coincide.

1.40. Show that the angle bisectors of a triangle meet at a point.

1.41. Show that the medians of a triangle meet at a point.

1.42. Prove that a subset of a linearly independent set of vectors is linearly independent.

1.43. Prove that two linearly independent vectors in E^2 form a basis in E^2.

1.44. Prove that three or more vectors in E^2 are linearly dependent.

1.45. Prove that if a_i, b_i, c_i are the components of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ with respect to a basis, then (i) $\mathbf{a} = \mathbf{b}$ iff $a_i = b_i$, (ii) $\mathbf{c} = \mathbf{a} + \mathbf{b}$ iff $c_i = a_i + b_i$, (iii) $\mathbf{b} = k\mathbf{a}$ iff $b_i = ka_i$.

1.46. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis. Determine whether $\mathbf{a} = \mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3$, $\mathbf{b} = \mathbf{u}_2 - \mathbf{u}_3$, $\mathbf{c} = 2\mathbf{u}_1 - \mathbf{u}_2 + 5\mathbf{u}_3$ are linearly independent. *Ans.* Yes

1.47. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and let $\mathbf{v}_1 = -\mathbf{u}_1 + \mathbf{u}_2 - \mathbf{u}_3$, $\mathbf{v}_2 = \mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3$, $\mathbf{v}_3 = 2\mathbf{u}_1 + \mathbf{u}_3$. Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis and find the components of $\mathbf{a} = 2\mathbf{u}_1 - \mathbf{u}_3$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
Ans. $\mathbf{u}_1 = -2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$, $\mathbf{u}_2 = 3\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3$, $\mathbf{u}_3 = 4\mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3$, $\mathbf{a} = -8\mathbf{v}_1 + 4\mathbf{v}_2 - 5\mathbf{v}_3$

1.48. Let $\mathbf{a} = -\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$. Find (*a*) $\mathbf{a} \cdot \mathbf{b}$, (*b*) $|\mathbf{a}|$, (*c*) $\cos \measuredangle(\mathbf{a}, \mathbf{b})$, (*d*) $P_{\mathbf{b}}(\mathbf{a})$, (*e*) $\mathbf{P}_{\mathbf{b}}(\mathbf{a})$.
Ans. (*a*) -4, (*b*) $\sqrt{6}$, (*c*) $-4/(3\sqrt{2})$, (*d*) $-4/\sqrt{3}$, (*e*) $-(4/3)(\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3)$

1.49. Find the direction cosines of the vector $\mathbf{a} = 2\mathbf{e}_1 + \mathbf{e}_2 - 3\mathbf{e}_3$. *Ans.* $2/\sqrt{14}$, $1/\sqrt{14}$, $-3/\sqrt{14}$

1.50. Determine x so that $\mathbf{a} = x\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{b} = 2\mathbf{e}_1 - x\mathbf{e}_2 + \mathbf{e}_3$ are orthogonal. *Ans.* $x = 1$

1.51. Factor $\alpha\gamma|\mathbf{a}|^2 - (\alpha\delta + \beta\gamma)(\mathbf{a} \cdot \mathbf{b}) + \beta\delta|\mathbf{b}|^2$. *Ans.* $(\alpha\mathbf{a} - \beta\mathbf{b}) \cdot (\gamma\mathbf{a} - \delta\mathbf{b})$

1.52. Let $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{b} = -\mathbf{e}_1 + 2\mathbf{e}_2 - 2\mathbf{e}_3$. Find a vector $\mathbf{c}$ so that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form the sides of a right triangle. *Ans.* $\mathbf{c} = \pm(2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3)$

1.53. Show that $\mathbf{g}_1 = (1/3)(2\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3)$, $\mathbf{g}_2 = (1/3)(\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3)$ and $\mathbf{g}_3 = (1/3)(2\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3)$ form an orthonormal basis and find $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in terms of $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$.
Ans. $\mathbf{e}_1 = (1/3)(2\mathbf{g}_1 + \mathbf{g}_2 + 2\mathbf{g}_3)$, $\mathbf{e}_2 = (1/3)(-2\mathbf{g}_1 + 2\mathbf{g}_2 + \mathbf{g}_3)$, $\mathbf{e}_3 = (1/3)(\mathbf{g}_1 + 2\mathbf{g}_2 - 2\mathbf{g}_3)$

1.54. Show that the sum of the squares of all the sides of a parallelogram is equal to the sum of the squares of its diagonals.

1.55. If $\mathbf{a} = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3$, $\mathbf{b} = 2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{c} = \mathbf{e}_2 + \mathbf{e}_3$, find (*a*) $\mathbf{a} \times \mathbf{b}$, (*b*) $\mathbf{b} \times \mathbf{a}$, (*c*) $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = [\mathbf{abc}]$, (*d*) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. *Ans.* (*a*) $5\mathbf{e}_1 + 7\mathbf{e}_2 + 3\mathbf{e}_3$, (*b*) $-5\mathbf{e}_1 - 7\mathbf{e}_2 - 3\mathbf{e}_3$, (*c*) 10, (*d*) $2\mathbf{e}_1 - 2\mathbf{e}_2 - 2\mathbf{e}_3$

1.56. Find a unit vector orthogonal to $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{b} = -\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$. *Ans.* $\pm(1/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_3)$

1.57. Find the distance d from the point P to the plane S where $\mathbf{a} = \mathbf{OP} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ is the vector from a point O on S to P and $\mathbf{b} = -\mathbf{e}_1 + \mathbf{e}_3$ and $\mathbf{c} = \mathbf{e}_1 - \mathbf{e}_2$ are along S. *Ans.* $d = |P_{\mathbf{b} \times \mathbf{c}}(\mathbf{a})|$

1.58. Prove that
$$[\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3][\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3] = \det\begin{pmatrix} \mathbf{u}_1 \cdot \mathbf{v}_1 & \mathbf{u}_1 \cdot \mathbf{v}_2 & \mathbf{u}_1 \cdot \mathbf{v}_3 \\ \mathbf{u}_2 \cdot \mathbf{v}_1 & \mathbf{u}_2 \cdot \mathbf{v}_2 & \mathbf{u}_2 \cdot \mathbf{v}_3 \\ \mathbf{u}_3 \cdot \mathbf{v}_1 & \mathbf{u}_3 \cdot \mathbf{v}_2 & \mathbf{u}_3 \cdot \mathbf{v}_3 \end{pmatrix}.$$

1.59. Prove that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d}) = 0$.

1.60. Prove that $[(\mathbf{a} \times \mathbf{b})(\mathbf{c} \times \mathbf{d})(\mathbf{e} \times \mathbf{f})] = [\mathbf{abd}][\mathbf{cef}] - [\mathbf{abc}][\mathbf{def}]$.

1.61. Show that if $\mathbf{a}$ and $\mathbf{b}$ lie in a plane normal to a plane containing $\mathbf{c}$ and $\mathbf{d}$, then $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 0$.

1.62. Let $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ be an arbitrary basis and let $\mathbf{v}_1 = \dfrac{\mathbf{u}_2 \times \mathbf{u}_3}{[\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3]}$, $\mathbf{v}_2 = \dfrac{\mathbf{u}_3 \times \mathbf{u}_1}{[\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3]}$, $\mathbf{v}_3 = \dfrac{\mathbf{u}_1 \times \mathbf{u}_2}{[\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3]}$. Show that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is dual to $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, i.e. $\mathbf{u}_i \cdot \mathbf{v}_j = \delta_{ij}$, $i, j = 1, 2, 3$.

1.63. Let $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ be dual bases. Show that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

1.64. Show that there exist two equivalent classes of oriented bases. Namely, show that if $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ do not have the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, then $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ have the same orientation. Thus we can say that two ordered bases have the same or opposite orientation.